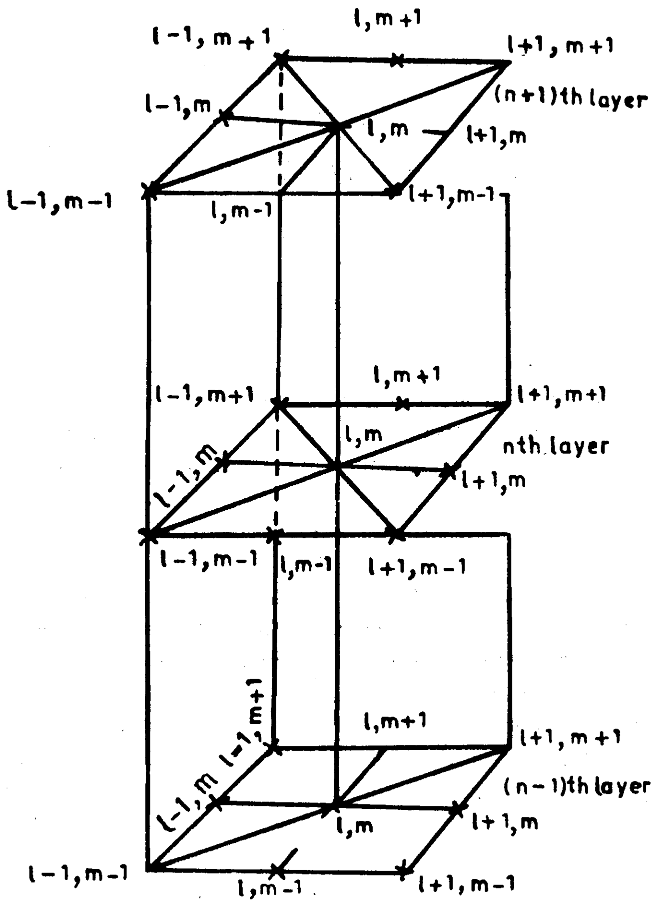

Fig. 5.7 Grid points for two space dimensions

5.6.1 Explicit difference schemes

Using (5.32) and (5.155), the explicit difference schemes for (5.152) can be written as

$$G(\nabla_t)u^{n+1}_{l,m} = r(\delta_x^2+\delta_y^2)u^n_{l,m} \tag{5.156}$$

where $u^n_{l,m}$ is an approximate value of $U^n_{l,m}$.
The most commonly used explicit scheme is obtained as the first approximation to (5.156). For example, the explicit difference scheme

$$\nabla_t u^{n+1}_{l,m} = r(\delta_x^2+\delta_y^2)u^n_{l,m} \tag{5.157}$$

or

$$u^{n+1}_{l,m} = (1-4r)u^n_{l,m}+r(u^n_{l-1,m}+u^n_{l+1,m}+u^n_{l,m-1}+u^n_{l,m+1})$$

has the order of accuracy $(k+h^2)$.

Using the Von-Neumann method of stability analysis, we look for solution $u^n_{l,m}$ of the form

$$u^n_{l,m} = A\xi^n \exp(i\theta_1 lh) \exp(i\theta_2 mh), \tag{5.158}$$

to the explicit difference equation (5.157). We find

$$\xi = 1-4r\left(\sin^2\frac{\theta_1 h}{2}+\sin^2\frac{\theta_2 h}{2}\right) \tag{5.159}$$

For stability $|\xi| \leqslant 1$ and so

$$-1 \leqslant 1-4r\left(\sin^2\frac{\theta_1 h}{2}+\sin^2\frac{\theta_2 h}{2}\right) \leqslant 1$$

Since $0 \leqslant \sin^2\theta_1 h/2, \sin^2\theta_2 h/2 \leqslant 1$, the stability condition is obtained as $0 < r \leqslant 1/4$. This method has the advantage of being simple and easy to apply. The restriction of small $k(k \leqslant h^2/4)$ usually requires a very large number of time steps. We can improve the stability condition if we write (5.157) as

$$u^{n+1}_{l,m} = (1+r\delta_x^2)(1+r\delta_y^2)u^n_{l,m} \tag{5.160}$$

The addition of the term $r^2\delta_x^2\delta_y^2u^n_{l,m}$ does not affect the order of accuracy of (5.157) as it is of higher order than the difference scheme. The formula (5.160) is of order $(k+h^2)$ and the stability condition is $0 < r \leqslant 1/2$. Thus the addition of the term $r^2\delta_x^2\delta_y^2u^n_{l,m}$ in (5.157) results in improving the stability requirements without any loss of the order of accuracy of the difference scheme. For $r = 1/6$, the difference scheme (5.160) has order of accuracy (k^2+h^4). In order to obtain the unconditionally stable explicit difference scheme we introduce the *Larkin* modification into (5.157). We consider the following formulas:

$$\nabla_t(1+r(\nabla_x+\nabla_y))u^{n+1}_{l,m} = r(\delta_x^2+\delta_y^2)u^n_{l,m} \tag{5.161}$$

and
$$\nabla_t(1-r(\Delta_x+\Delta_y))u^{n+1}_{l,m} = r(\delta_x^2+\delta_y^2)u^n_{l,m} \tag{5.162}$$

substituting (5.158) into (5.161) and simplifying, we obtain

$$\xi = \frac{1-2r+r(\exp(i\theta_1 h)+\exp(i\theta_2 h))}{1+2r-r(\exp(-i\theta_1 h)+\exp(-i\theta_2 h))} \tag{5.163}$$

We denote the numerator and denominator of the right side of (5.163) by N and D respectively. The imaginary parts of N and D are identical. Thus the absolute value of N/D will be less than or equal to unity provided the following condition is satisfied,

$$[Re(D)]^2-[Re(N)]^2 \geqslant 0 \tag{5.164}$$

By expansion of N and D it follows that

$$[Re(D)]^2-[Re(N)]^2 = 4r(2-\cos\theta_1 h-\cos\theta_2 h)$$

This obviously satisfies (5.164) for $r > 0$. Thus the difference scheme given by (5.161) is unconditionally stable. Performing similar analysis on (5.162) we find that (5.162) is also unconditionally stable. The truncation error of the formula (5.161) is given by

$$T^n_{l,m} = \nabla_t[1+r(\nabla_x+\nabla_y)]U^{n+1}_{l,m} - r(\delta^2_x+\delta^2_y)U^n_{l,m} \tag{5.165}$$

Expanding each term on the right-hand side of (5.165) in the Taylor series about (lh, mh, nk) and using (5.162) we get

$$k^{-1}T^n_{l,m} = \frac{k}{h}\left(\frac{\partial^2 U^n_{l,m}}{\partial x\,\partial t}+\frac{\partial^2 U^n_{l,m}}{\partial y\,\partial t}\right)+\frac{k^2}{2h}\left(\frac{\partial^3 U^n_{l,m}}{\partial x\,\partial t^2}+\frac{\partial^3 U^n_{l,m}}{\partial y\,\partial t^2}\right)+0(k^2+h^2) \tag{5.166}$$

which will tend to zero if $(k/h) \to 0$ as $h \to 0$. By diminishing h and k such that r is constant, the leading term of (5.166) would be of first degree in h, all other terms being of higher order. By a similar procedure, the truncation error of (5.162) is obtained as

$$k^{-1}T^n_{l,m} = -\frac{k}{h}\left(\frac{\partial^2 U^n_{l,m}}{\partial x\,\partial t}+\frac{\partial^2 U^n_{l,m}}{\partial y\,\partial t}\right)-\frac{1}{2}\frac{k^2}{h}\left(\frac{\partial^3 U^n_{l,m}}{\partial x\,\partial t^2}+\frac{\partial^3 U^n_{l,m}}{\partial y\,\partial t^2}\right)+0(k^2+h^2) \tag{5.167}$$

The leading terms of (5.166) and (5.167) have opposite signs. Averaging the results of the schemes (5.161) and (5.162), we get difference methods which have truncation error of $0(k^2+h^2)$.

Let $u^{*n}_{l,m}$ and $u^{**n}_{l,m}$ be the solution of the difference schemes (5.161) and (5.162) respectively. Furthermore, let the solutions $u^{*n}_{l,m}$ and $u^{**n}_{l,m}$ also satisfy the initial and boundary conditions (5.153). Then, the solution $u^{n+1}_{l,m}$ at any time level $(n+1)$ may be given by

$$u^{n+1}_{l,m} = \frac{1}{2}\,(u^{*n+1}_{l,m}+u^{**n+1}_{l,m}) \tag{5.168}$$

The values $u^{*n+1}_{l,m}$ can be calculated explicitly using (5.161). In this case calculations proceed from the nodal point nearest to the boundaries $x = 0$ and $y = 0$ in a sequence of increasing l and m. The needed values of $u^{n+1}_{l-1,m}$, $u^n_{l,m}$ and $u^n_{l+1,m}$ will be known. In a similar manner $u^{**n+1}_{l,m}$ can be calculated explicitly from (5.162) beginning at the boundaries $x = 1$, $y = 1$ moving in a sequence of decreasing l and m. In equations (5.161) and (5.162) the values $u^{*n+1}_{l,m}$ and $u^{**n+1}_{l,m}$ for one step become $u^{*n}_{l,m}$ and $u^{**n}_{l,m}$ for the next step. These results now may be used in (5.168) to compute $u^{n+1}_{l,m}$.

The *DuFort-Frankel* scheme for two space dimensions can be written as

$$\left(\nabla_t - \frac{1}{2}\nabla^2_t\right)u^{n+1}_{l,m} = r(\delta^2_x+\delta^2_y-2\delta^2_t)\,u^n_{l,m} \tag{5.169}$$

This is an unconditionally stable scheme and the truncation error is given by

$$k^{-1}T^n_{l,m} = 0\left(k^2+h^2+\left(\frac{k}{h}\right)^2\right) \tag{5.170}$$

which tends to zero if $k/h \to 0$ as $h \to 0$, $k \to 0$. In order to apply the scheme (5.169) to (5.152) we take $u^0_{l,m}$ from the initial conditions and the values $u^1_{l,m}$ are generally obtained from a two level difference scheme.

Example 5.4 Solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

$$u(x, y, 0) = \cos\frac{\pi x}{2}\cos\frac{\pi y}{2} \qquad -1 \leqslant x, y \leqslant 1, t = 0$$

$$u = 0 \qquad x = \pm 1, y = \pm 1, t > 0$$

using the second order method

$$u^{n+1}_{l,m} = (1-4r)u^n_{l,m}+r(u^n_{l+1,m}+u^n_{l-1,m}+u^n_{l,m+1}+u^n_{l,m-1})$$

with $h = \frac{1}{2}$ and $r = \frac{1}{6}$.

The grid points are

$$\begin{aligned} x_l &= \pm\, lh, & 0 \leqslant l \leqslant M+1, \quad & h = 1/(M+1)\\ y_m &= \pm\, mh, & 0 \leqslant m \leqslant M+1, \quad & h = 1/(M+1)\\ t_n &= nk, & n = 0, 1, 2, \ldots, \quad & k > 0 \end{aligned}$$

On account of symmetry, we need only consider one eighth of the square. The initial and boundary conditions become

$$u^0_{l,m} = \cos\frac{l\pi h}{2}\cos\frac{m\pi h}{2}$$

$$u^n_{M+1,0} = u^n_{0,M+1} = 0$$

where $M = 1$

For $r = \frac{1}{6}$, the difference method becomes

$$u^{n+1}_{l,m} = \frac{1}{6}(u^n_{l+1,m}+u^n_{l-1,m}+u^n_{l,m+1}+u^n_{l,m-1}+2u^n_{l,m})$$

We have

$$n = 0, \qquad u^1_{l,m} = \frac{1}{6}(u^0_{l+1,m}+u^0_{l-1,m}+u^0_{l,m+1}+u^0_{l,m-1}+2u^0_{l,m})$$

$$l = 0, m = 0, u^1_{0,0} = \frac{1}{6}(u^0_{1,0}+u^0_{-1,0}+u^0_{0,1}+u^0_{0,-1}+2u^0_{0,0})$$